$$F_s = A_s p_s \sin\theta[\sin\theta + \tfrac{2}{3}] \quad (17)$$

taking into account the diffusely reflecting surface. The resulting precession torque T_p is zero since the axial forces given by Eq. (14) and Eq. (16) are equal; however, the spin torque (which will be used to oppose despin) is equal to

$$T_s = \tfrac{2}{3}A_s p_s R_s \sin\theta \quad (18)$$

provided care is taken to make the IR emissivities of both black and white sides equal.

Black and white surfaces of the forementioned type may be used with corner mirrors, such as shown in Fig. 2, without affecting the precession torque of the latter but in a manner such as to oppose the despin torque. If this is done, the expression for the despin torque of the combination is the difference between Eqs. (9) and (18) or

$$T_s = p_s \sin\theta[4RA \sin\theta - \tfrac{2}{3}A_s R_s] \quad (19)$$

For

$$\sin\theta = \tfrac{1}{6}[A_s R_s/AR] \quad (20)$$

there will be neither a spin-up nor a despin torque, but a spin-up torque for smaller values of θ (when the spin rate is also small) and a despin torque for larger values of θ (when the spin rate is higher). Equation (20), together with Eq. (12), gives the asymptotic values of both angular momentum and tracking error, although the asymptotic tracking error is determined by Eq. (20) alone. The smaller the relative size of the spin regulating array, the smaller this tracking error. The array of black and white surfaces may be radial extensions of the long side of the corner mirrors but, if the latter are close together, mutual interference may be avoided by using a smaller number of larger surfaces, one surface being sufficient. Control of the asymptotic spin rate is also possible by adjusting the angle of attack of the spin regulating surfaces by remote control or by centrifugal force. A photograph of an attitude stabilizing array of corner mirrors having spin regulating surfaces is shown in Fig. 5.

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Singular Extremals in Lawden's Problem of Optimal Rocket Flight

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The problem of optimal rocket flight in an inverse square law force field has been studied extensively by Lawden and Leitmann. Periods of zero thrust, intermediate thrust, and maximum thrust are possible subarcs of the solution according to analysis of the Euler-Lagrange equations and the Weierstrass necessary condition. Arcs of intermediate thrust have been examined recently by Lawden; however, the question of whether or not such arcs actually may furnish a minimum has been left unresolved. The present paper derives the singular extremals of Lawden's problem by means of the Legendre-Clebsch necessary condition applied in a transformed system of state and control variables. These are obtained as circular orbits along which the thrust is zero and intermediate thrust arcs are found in Lawden's analysis. Since these solutions satisfy only the weak form of the Legendre-Clebsch condition, i.e., the extremals are singular in the transformed system of variables, the question of their minimality remains unanswered.

Introduction

The problem of optimal rocket flight in an inverse square law force field has been investigated by Lawden[1, 2] and Leitmann.[3] Although considerable progress has been made in the study of properties of the solution, a question remains as to the possible appearance of subarcs of intermediate thrust.[4, 5] Such arcs are among the *singular* extremals of the problem, in classical variational terminology, and are resistant to analytical efforts owing to the unavailability of a general theory applicable to singular cases.[6]

This paper first presents a brief development of the Euler-Lagrange equations and the Weierstrass necessary condition along the lines of previous investigations and then proceeds to an analysis of intermediate thrust arcs.

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Lawden's Problem

The equations of motion for a rocket in two-dimensional flight are given by

$$\dot{u} = (T/m)\sin\theta + Y \quad (1)$$

$$\dot{v} = (T/m)\cos\theta + X \tag{2}$$

$$\dot{y} = u \tag{3}$$

$$\dot{x} = v \tag{4}$$

$$\dot{m} = -(T/c) \tag{5}$$

Here y and x, Cartesian coordinates in an inertial frame, u and v, the corresponding velocity components, and the mass m are the state variables of the problem. The control variables are the thrust magnitude T and the thrust direction angle θ. The former is subject to a constraint of inequality type

$$0 \leq T \leq \bar{T} \tag{6}$$

corresponding to an assumed capability of throttling the rocket motor over a thrust range of zero to maximum thrust $\bar{T}$. The gravitational force components Y and X are functions of the position coordinates and, in the most general case, of time as well. The present analysis will be concerned with the case of an inverse square law gravitational field.

Stated in the Mayer form, the optimal rocket flight problem is to determine a solution of Eqs. (1–6), subject to appropriate boundary conditions, which furnishes a minimum of a function P of the state variable terminal values and the terminal time. In terms of the generalized Hamiltonian function

$$H = \lambda_u[(T/m)\sin\theta + Y] + \lambda_v[(T/m)\cos\theta + X] + \lambda_y u + \lambda_x v + \lambda_m(-T/c) \tag{7}$$

The Euler-Lagrange equations for the problem are given as

$$\dot{\lambda}_u = -(\partial H/\partial u) = -\lambda_y \tag{8}$$

$$\dot{\lambda}_v = -(\partial H/\partial v) = -\lambda_x \tag{9}$$

$$\dot{\lambda}_y = -(\partial H/\partial y) = -[\lambda_u(\partial Y/\partial y) + \lambda_v(\partial X/\partial y)] \tag{10}$$

$$\dot{\lambda}_x = -(\partial H/\partial x) = -[\lambda_u(\partial Y/\partial x) + \lambda_v(\partial X/\partial x)] \tag{11}$$

$$\dot{\lambda}_m = -(\partial H/\partial m) = (T/m^2)(\lambda_u \sin\theta + \lambda_v\cos\theta) \tag{12}$$

in which the functions λ_u, λ_v, λ_y, λ_x, and λ_m are the usual Lagrange multipliers.

The control variables T and θ must satisfy the relation

$$H(T^*,\theta^*) \geq H(T,\theta) \tag{13}$$

for all admissible T^*, θ^*; which is to say that T and θ provide a minimum of the function H, subject to Eq. (6). This is the extended form of the Weierstrass necessary condition as derived by Pontryagin et al.[7]

A minimum of H is attained for

$$\sin\theta = \frac{-\lambda_u}{(\lambda_u^2 + \lambda_v^2)^{1/2}} \qquad \cos\theta = \frac{-\lambda_v}{(\lambda_u^2 + \lambda_v^2)^{1/2}} \tag{14}$$

$$T = 0 \text{ for } \rho \equiv -(1/m)(\lambda_u^2 + \lambda_v^2)^{1/2} - (\lambda_m/c) > 0 \tag{15}$$

$$T = \bar{T} \text{ for } \rho \equiv -(1/m)(\lambda_u^2 + \lambda_v^2)^{1/2} - (\lambda_m/c) < 0 \tag{16}$$

In the case $\rho = 0$ the function H is not explicitly dependent upon T and the thrust magnitude is not determined by the Weierstrass necessary condition.

Portions of a solution of the Euler-Lagrange equations for which ρ vanishes identically, i.e., over a finite time interval, are known as *singular* subarcs according to the terminology of the classical theory, the criterion being the vanishing of the determinant

$$\begin{vmatrix} \partial^2 H/\partial T^2 & \partial^2 H/\partial\theta\partial T \\ \partial^2 H/\partial T\partial\theta & \partial^2 H/\partial\theta^2 \end{vmatrix} \tag{17}$$

This definition applies only if the function H is stationary at its minimum

$$\partial H/\partial\theta = \partial H/\partial T = 0 \tag{18}$$

Intermediate Thrust Subarcs

The possible appearance of singular subarcs in a problem is accompanied by considerable analytical difficulty. There is no powerful general method available for determining these subarcs, or whether they may furnish a minimum even in the local sense, i.e., over short intervals, or in what manner they form segments of the minimizing arc. Valuable insight into these questions is provided by the Green's Theorem method of Miele,[8] which, however, is restricted severely in number of variables, as regards its applicability.

For the problem presently considered, Lawden has examined arcs of intermediate thrust satisfying the Euler-Lagrange equations and the Weierstrass necessary condition.[4, 5] His results indicate the existence of a family of such intermediate thrust arcs, including a spiral corresponding to the case $H = 0$, analyzed in some detail. The question of whether or not such arcs actually may furnish a minimum, however, has been left unresolved. In the present analysis an alternate approach to the intermediate thrust arcs is pursued.

New variables ψ, β, γ, ϕ, V are introduced now replacing u, v, y, x, m according to the transformation

$$\psi = y\sin\theta + x\cos\theta + (1/\omega)(u\cos\theta - v\sin\theta) \tag{19}$$

$$\beta = c\ln m + u\sin\theta + v\cos\theta \tag{20}$$

$$\gamma = y\cos\theta - x\sin\theta \tag{21}$$

$$\phi = y\sin\theta + x\cos\theta \tag{22}$$

$$V = -c\ln m \tag{23}$$

It is verified readily that this transformation is nonsingular by the nonvanishing of the Jacobian determinant

$$\Delta = \frac{\partial(\psi, \beta, \gamma, \phi, V)}{\partial(u, v, y, x, m)} = -\frac{c}{\omega m} \neq 0 \tag{24}$$

By a formal process the equations of state in the new system of variables are obtained as

$$\dot{\psi} = \gamma\omega + (1/\omega)(Y\cos\theta - X\sin\theta) - (\mu/\omega)(\psi - \phi) \tag{25}$$

$$\dot{\beta} = Y\sin\theta + X\cos\theta + \omega^2(\psi - \phi) \tag{26}$$

$$\dot{\gamma} = \omega(\psi - 2\phi) \tag{27}$$

$$\dot{\phi} = \beta + \gamma\omega + V \tag{28}$$

$$\dot{V} = T/m = e^{V/c}\,T \tag{29}$$

$$\dot{\theta} = \omega \tag{30}$$

$$\dot{\omega} = \mu \tag{31}$$

It has been assumed tacitly in the course of the manipulations leading to Eqs. (25–31) that the steering angle θ is twice differentiable, i.e., that the derivatives $\dot{\theta} = \omega$ and $\ddot{\theta} = \mu$ exist. Examination of Eqs. (8–14) indicates that such an assumption is justified if the gravitational force components Y and X possess first partial derivatives, except for a finite number of points along the trajectory, corresponding to thrust direction reversals, at which λ_u and λ_v vanish simultaneously. Such reversal points are excluded from the segments of arc analyzed in the following.

It would appear upon first inspection of the Eqs. (25–31) governing the new variables that an unwarranted increase in complexity has been realized. Our motivation becomes clear, however, when it is observed that the variables T and V appear *only* in Eqs. (28) and (29), and that as a consequence of this, the multipliers λ_ϕ and λ_V vanish along the singular subarcs. Means of synthesizing transformations having this property will be discussed in another paper presently in preparation.

Consider a segment of arc of intermediate thrust, i.e., over which the strict inequality in (6) holds

$$0 < T < \bar{T} \tag{32}$$

it follows that neighboring thrust programs

$$T + \delta T = T(t) + \epsilon\eta(t) \tag{33}$$

will also satisfy Eq. (32) if the magnitude ϵ of the (otherwise arbitrary) thrust variation is taken sufficiently small.

Evidently if attention is restricted to small variations in T, V, and ϕ, the variable ϕ may be regarded as a control variable over an intermediate thrust arc, as implied by the vanishing of the multipliers λ_ϕ and λ_V. It is noted that the coefficient of T in Eq. (29) and the coefficient of V in Eq. (28) never vanish, and accordingly, that an admissible variation in thrust δT may be found which produces an approximation as close as one wishes to an arbitrary variation $\delta\phi(t)$, provided that the magnitude of $\delta\phi$ is sufficiently small. With ϕ in the role of control variable and small variations assumed, the intermediate thrust arcs must satisfy the necessary conditions for a weak relative minimum.

The Euler-Lagrange equations for the system (25–27, 30, and 31) are

$$\dot{\lambda}_\psi = -(\partial H/\partial\psi) = \lambda_\psi(\mu/\omega) - \lambda_\beta\omega^2 - \lambda_\gamma\omega \tag{34}$$

$$\dot{\lambda}_\beta = -(\partial H/\partial\beta) = 0 \tag{35}$$

$$\dot{\lambda}_\gamma = -(\partial H/\partial\gamma) = -\lambda_\psi[\omega + (1/\omega)(\partial/\partial\gamma)(Y\cos\theta - X\sin\theta)] - \lambda_\beta(\partial/\partial\gamma)(Y\sin\theta + X\cos\theta) \tag{36}$$

$$\dot{\lambda}_\theta = -(\partial H/\partial\theta) = -\lambda_\psi(\partial/\partial\theta)(Y\cos\theta - X\sin\theta) - \lambda_\beta(\partial/\partial\theta)(Y\sin\theta + X\cos\theta) \tag{37}$$

$$\dot{\lambda}_\omega = -(\partial H/\partial\omega) = -\lambda_\psi[\gamma - (1/\omega^2)(Y\cos\theta - X\sin\theta) + (\mu/\omega^2)(\psi - \phi)] - 2\lambda_\beta\omega(\psi - \phi) - \lambda_\gamma(\psi - 2\phi) - \lambda_\theta \tag{38}$$

$$\partial H/\partial\phi = \lambda_\psi[(\mu/\omega) + (1/\omega)(\partial/\partial\phi)(Y\cos\theta - X\sin\theta)] + \lambda_\beta[-\omega^2 + (\partial/\partial\phi)(Y\sin\theta + X\cos\theta)] - 2\lambda_\gamma\omega = 0 \tag{39}$$

$$\partial H/\partial\mu = -\lambda_\psi[(\psi - \phi)/\omega] + \lambda_\omega = 0 \tag{40}$$

The Legendre-Clebsch necessary condition for a weak relative minimum is

$$\frac{\partial^2 H}{\partial\phi^2}\,\delta\phi^2 + 2\,\frac{\partial^2 H}{\partial\phi\partial\mu}\,\delta\phi\delta\mu + \frac{\partial^2 H}{\partial\mu^2}\,\delta\mu^2 \geq 0 \tag{41}$$

for arbitrary $\delta\phi$, $\delta\mu$. Positive semidefiniteness of this quadratic form requires that

$$(\partial^2 H/\partial\phi^2) \geq 0 \tag{42}$$

$$(\partial^2 H/\partial\mu^2) \geq 0 \tag{43}$$

$$(\partial^2 H/\partial\phi^2)(\partial^2 H/\partial\mu^2) - (\partial^2 H/\partial\phi\partial\mu)^2 \geq 0 \tag{44}$$

There is

$$\partial^2 H/\partial\phi^2 = \lambda_\beta(\partial^2/\partial\phi^2)(Y\sin\theta + X\cos\theta) + (\lambda_\psi/\omega)(\partial^2/\partial\phi^2)(Y\cos\theta - X\sin\theta) \tag{45}$$

$$\partial^2 H/\partial\mu^2 = 0 \tag{46}$$

$$\partial^2 H/\partial\phi\partial\mu = \lambda_\psi/\omega \tag{47}$$

From Eqs. (42–44) and Eqs. (45–47) it follows that $\lambda_\psi = 0$.

With this simplification and the elimination of the multiplier variables from the Euler-Lagrange equations (34–40), one arrives at

$$\omega^2 + (\partial Z/\partial\phi) = 0 \tag{48}$$

$$-\mu + (\partial Z/\partial\gamma) = 0 \tag{49}$$

$$\mu\phi + \omega^2\gamma - [x(\partial/\partial y) - y(\partial/\partial x)]Z = 0 \tag{50}$$

in which

$$Z \equiv Y\sin\theta + X\cos\theta \tag{51}$$

is the component of gravitational force along the thrust direction.

In the case of an inverse square law gravitational field

$$Y = \frac{-ky}{(x^2 + y^2)^{3/2}}, \qquad X = \frac{-kx}{(x^2 + y^2)^{3/2}} \tag{52}$$

Eqs. (48, 49, and 50) become

$$\omega^2 + [k(2\phi^2 - \gamma^2)/(\phi^2 + \gamma^2)^{5/2}] = 0 \tag{53}$$

$$-\mu + [3k\phi\gamma/(\phi^2 + \gamma^2)^{5/2}] = 0 \tag{54}$$

$$\mu\phi + \{\omega^2 - [k/(\phi^2 + \gamma^2)^{3/2}]\}\gamma = 0 \tag{55}$$

If ω is eliminated between Eqs. (53) and (55), one obtains

$$\phi\{\mu - [3\gamma k/(\phi^2 + \gamma^2)^{5/2}]\} = 0 \tag{56}$$

The vanishing of the first factor $\phi = 0$, circumferential thrust, leads to $\mu = 0$, $\omega =$ constant, and

$$|\omega| = k/r^{3/2} \tag{57}$$

where $r = (\phi^2 + \gamma^2)^{1/2}$ is the radius. This is the orbital frequency for free fall circular motion. The vanishing of the second factor indicates that Eq. (55) is satisfied identically along solutions of Eqs. (53) and (54), which are the equations of Lawden's intermediate thrust solutions (Ref. 5), although in rather different notation.

Concluding Remarks

The present analysis amounts to little more than an alternate derivation of Lawden's results on intermediate thrust subarcs, similarly inconclusive on the question of minimality. This is a result of the singular extremals of the original problem being also singular in the transformed system of variables, i.e., only the weak form of the Legendre-Clebsch condition in these variables is met. The most suggestive feature of the analysis is the vanishing of two of the multipliers associated with the new variables. This would seem to indicate a possibility of dealing with the problem in a state space of reduced dimension.

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